All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $F\colon \mathbf{R}^2 \to \mathbf{R}$ be the function $F(x,y) = xy^2 + x - x^3$. Find the critical points of F, find the Hessian of F at each critical point, and classify the critical points as maxima, minima or saddle points.

   (b) Let $P(x,y,z) = xyz$ and consider its restriction to the ellipsoid $G(x,y,z) = 0$ where
$$G(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$
   for some constants $0 < a < b < c$.

   (i) Show that the critical points of P restricted to the ellipsoid are the six points
$$(\pm a, 0, 0),\ (0, \pm b, 0),\ (0, 0, \pm c)$$
   and the eight points
$$\frac{1}{\sqrt{3}}(\pm a, \pm b, \pm c)$$
   where the $\pm$ signs in the latter case are independent of each other.

   (ii) Which of these critical points are already critical points of P before restriction to the ellipsoid and which are not? In the latter case, give a geometric interpretation of what is happening in terms of the level sets of P and of G.

2. Consider the quasilinear first-order partial differential equation for the function $\phi(x, y)$:

$$x^2 \frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial y} + y = 0.$$

(a) Write down the characteristic vector field and show that its integral curves are

$$(x(t), y(t), z(t)) = \left(-\frac{1}{t+K}, M \cos t, -M \sin t \right)$$

where K and M are constants of integration.

(b) If we impose the initial condition $\phi(1, s) = s$, show that the solution surface can be parametrised as

$$(s, t) \mapsto \left(-\frac{1}{t - 1 + \frac{\pi}{4}}, s\sqrt{2} \cos t, -s\sqrt{2} \sin t \right). \qquad (*)$$

(c) The caustic of a surface

$$(s, t) \mapsto (x(s, t), y(s, t), z(s, t))$$

is the set of values (x, y) where

$$\det \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = 0.$$

What is the caustic of the solution surface (*)?

(d) Find a function $\phi(x, y)$, defined on a subset of $\mathbf{R}^2$, such that the graph of ϕ coincides with a part of the solution surface. Remember to specify the domain of definition of your function.

3. Suppose that a function $y(x)$ satisfies the Euler-Lagrange equation for a functional

$$\int_a^b L(y, y')dx$$

such that the Lagrangian $L = L(y, y')$ has no explicit dependence on the variable x. Prove the *Beltrami identity*:

$$L - y'\frac{\partial L}{\partial y'} = \text{constant}.$$

Consider the space of functions $y(x)$

$$X = \{y \colon [0,1] \to \mathbf{R} \; : \; y(0) = 0, \; y(1) = 0\}.$$

Define the functionals

$$F(y) = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

and

$$G(y) = \int_0^1 (y - 1)dx.$$

By extremising a suitably modified functional, show that if $y(x)$ is a critical point of F restricted to the set $\{G(y) = 0\}$ then the graph of y is a segment of a circle. *You do not need to compute any of the constants of integration or the Lagrange multiplier.*

What would the graph of a critical point be if we did not impose the restriction $G(y) = 0$?

4. (a) Suppose that $\phi \colon [0, \pi] \times [0, \pi] \to \mathbf{R}$ solves Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

By separating variables and solving the resulting ordinary differential equations, derive expressions for the separated solutions $\phi(x, y) = X(x)Y(y)$.

(b) We impose the boundary conditions

$$\phi(x, 0) = 0, \; \phi(x, \pi) = e^x - 1, \; \phi(0, y) = 0, \; \phi(\pi, y) = e^y - 1.$$

Show that the solution ϕ is

$$\phi(x, y) = \frac{xy(e^\pi - 1)}{\pi^2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{e^\pi(-1)^n - 1}{n(n^2 + 1)} \left(\sin nx \frac{\sinh ny}{\sinh n\pi} + \frac{\sinh nx}{\sinh n\pi} \sin ny \right).$$

Hint: It may help you when computing Fourier series to remember that

$$\sin(nx) = \frac{1}{2i}(e^{inx} - e^{-inx}).$$

5. What does it mean for a quasilinear partial differential equation of second order to be elliptic, parabolic or hyperbolic? Verify that the minimal surface equation

$$\frac{\partial^2\varphi}{\partial x^2}\left(1+\left(\frac{\partial\varphi}{\partial y}\right)^2\right)+\frac{\partial^2\varphi}{\partial y^2}\left(1+\left(\frac{\partial\varphi}{\partial x}\right)^2\right)=2\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial x\partial y}$$

is elliptic.

Consider the wave equation

$$\frac{\partial^2\phi}{\partial t^2}=c(x)^2\frac{\partial^2\phi}{\partial x^2},\quad x\in\mathbf{R},\ t\in\mathbf{R},$$

where $c(x)\neq 0$ may depend on x. Show that this equation is hyperbolic.

Suppose that c in the wave equation depends discontinuously on x:

$$c(x)=\begin{cases}1 & \text{if } x<0\\ 2 & \text{if } x\geqslant 0\end{cases}$$

Find the separated solutions $\phi(x,t)=X(x)T(t)$ to this equation which are oscillatory (i.e. trigonometric) in time and such that $\phi(x,t)$ and $\partial\phi/\partial x(x,t)$ are continuous at $x=0$.

Suppose that $\phi(x,t)=\sin x\sin t$ for $x<0$. What is $\phi(x,t)$ for $x>0$?

6. State coordinates u and v such that

$$3\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y^2}=C\frac{\partial^2\phi}{\partial u\partial v}\qquad(1)$$

for some constant C, and find the constant C. Check explicitly that Equation (1) holds for the coordinates you have chosen.

Sketch the lines $u=$ constant and $v=$ constant on a spacetime diagram.

Find the general solution of the equation

$$3\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y^2}=\frac{16}{9}(x^2-2xy-3y^2).$$

If we are given that

$$\phi(x,0)=\cos x,\quad \frac{\partial\phi}{\partial y}(x,0)=e^x,$$

what is ϕ?

END OF PAPER